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Interval Monte Carlo methods for structural reliability

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ABSTRACT

This paper considers structural reliability assessment when statistical parameters of distribution functions can not be determined precisely due to epistemic uncertainty. Uncertainties in parameter estimates are modeled by interval bounds constructed from confidence intervals. Reliability analysis needs to consider families of distributions whose parameters are within the intervals. Consequently, the probability of failure will vary in an interval itself. To estimate the interval failure probability, an interval Monte Carlo method has been developed which combines simulation process with the interval analysis. In this method, epistemic uncertainty and aleatory uncertainty are propagated separately through finite element-based reliability analysis. Interval finite element method is utilized to model the ranges of structural responses accurately. Examples are presented to compare the interval estimates of limit state probability obtained from the proposed method and the Bayesian approach.

1. Introduction

A major step in structural reliability analysis is the modeling and quantification of various sources of uncertainty. It is common in engineering practice to distinguish between aleatory uncertainty and epistemic uncertainty [1,2]. Aleatory uncertainty is due to the inherent random nature of physical quantities (e.g., variabilities in yield strength of steel). Aleatory uncertainties are generally modeled by random variables. In contrast to aleatory uncertainty, epistemic uncertainty is knowledge-based, and arises from imperfect modeling, simplification and limited availability of database. Possible sources of epistemic uncertainty include model uncertainty and statistical uncertainty. Model uncertainty is related to the discrepancy between real structural behavior and its simplified representation in mathematical models such as finite element (FE) models. Statistical uncertainty is another important source of epistemic uncertainty. The probability distribution to describe a random phenomenon is generally not precisely known. The statistical parameters (e.g., mean and standard deviation) are usually estimated by statistical inference from sampled observational data and a point estimator is used to approximate the 'true' parameter. Thus the distribution is itself subject to some uncertainty. Statistical uncertainty may be significant if only a limited sample of data is available. While the model uncertainty is beyond the scope of this work, the statistical uncertainty will be considered here. In particular, we are interested in reliability assessment when the parameters of distribution functions can not be determined precisely due to small sample size.

The Bayesian approach is routinely used to consider the uncertainties associated with estimation of parameters of a probability distribution. The unknown parameters are assumed to be (Bayesian) random variables [3]. The epistemic uncertainty and aleatory uncertainty are combined through the total probability theorem. With the Bayesian approach, subjective judgments are required to estimate the Bayesian random variables. The estimate of the Bayesian random variables can be improved when additional data become available. Before receiving more data, however, the Bayesian approach remains a subjective representation of uncertainty.

In this paper, incomplete knowledge of the distribution parameters is modeled by the interval bounds constructed from confidence intervals. Based on the observational data, a confidence interval is established over which the parameter is located at a specified level of confidence [3]. The epistemic uncertainties in estimating the parameters are reflected in the widths of the intervals. With the interval approach, reliability assessment needs to consider families of distributions whose parameters are within the intervals. One practical way to describe the ensemble of distributions is to specify its lower and upper bounds. The mathematical frameworks using this methodology include Dempster-Shafer evidence theory [4], random set theory [5] and probability boxes [6]. The computation procedures are typically a combination of interval analysis and the Cartesian product method [7]. Lower and

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upper bounds on the limit state probability are computed. Penmetsa and Grandhi [8] considered random variables and interval variables simultaneously in structural reliability analysis. Function approximation was used to reduce the number of simulations. In [6,9], probability boxes and Dempster-Shafer structures were used to bound imprecisely specified probability distributions. The method was applied to environmental risk assessment. Tonon et al. [10] considered the reliability analysis for an aircraft wing at the early stage of design process where the observational data is not point-valued but set-valued. Random set theory was employed to represent the envelope of all probability distributions compatible with the available information. In [11], random sets were constructed from limited observational data by applying inequalities of Tchebycheff to the sample mean. The method was utilized to bound the statistics of the displacement response of a cantilever sheet pile wall. In [12], interval analysis was combined with first-order reliability method (FORM). The unknown means and standard deviations of random variables were modeled as interval numbers. Interval analysis was applied to the closed-form solutions of FORM, and the bounds on structural system reliability index were evaluated.

Despite the research progress, the computing effort, especially when Cartesian product method is used, is a barrier to the practical application of non-traditional uncertainty models. The issue of computational cost becomes more serious when the reliability analysis is FE-based, i.e., the structural responses are obtained through FE analyses.

This paper proposes an interval Monte Carlo method to propagate interval parameters through FE-based reliability assessment. The FE portion of the analysis is carried out by the authors' recently developed interval finite element method (FEM). An interval estimate on the failure probability is computed. Two plane structures are analyzed to illustrate the proposed method. The interval failure probabilities obtained from the proposed method and the Bayesian approach are compared through the examples.

2. Reliability analysis under parameter uncertainty

The basic reliability problem is defined by the multiple integration

$$p_f = P(G(\mathbf{X}) \leqslant 0) = \int \cdots \int_{G(\mathbf{X})\leqslant 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}. \tag{1}$$

Here p_f represents the probability of failure of the structure. $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ is the n-dimensional vector of the basic random variables representing uncertain quantities such as applied loads, material strength and stiffness. $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for $\mathbf{X}$. $G(\mathbf{X})$ is the limit state function and $G(\mathbf{X}) \leqslant 0$ defines the failure state.

Let θ denote the (unknown) statistical parameters of the distribution function $f_{\mathbf{X}}(\mathbf{x})$. In the Bayesian approach θ are modeled to be random variables, thus $f_{\mathbf{X}}(\mathbf{x})$ becomes a conditional distribution function $f_{\mathbf{X}|\theta}(\mathbf{x}|\theta)$. Clearly the presence of θ implies that the probability p_f is random itself. The expectation of the conditional failure probability is often computed to characterize the total uncertainty [1,13]

$$\tilde{p_f} = \int_\theta p_f(\theta) f_\theta(\theta)d\theta = \int \cdots \int_{G(\mathbf{X},\theta)\leqslant 0} f_{\mathbf{X}|\theta}(\mathbf{x}|\theta) f_\theta(\theta)d\mathbf{x}d\theta \tag{2}$$

where $f_\theta(\theta)$ is the joint probability distribution function of the parameters θ. However, there is a perception that the mean (or other point estimate) of p_f does not fully characterize the epistemic uncertainty in the failure probability [14,15]. Alternatively, epistemic and aleatory uncertainties can be propagated through reliability analysis separately to obtain an interval estimate of p_f. Within the framework of the Bayesian approach, one can compute the frequency distribution of p_f, based on which a Bayesian confidence interval on p_f can be estimated. Interval estimate of failure probability can provide useful information to decision-makers about the variability in reliability or risk [14]. When applying the Bayesian approach, subjective judgment is needed to estimate $f_\theta(\theta)$. The estimate of $f_\theta(\theta)$ can be improved by using the Bayesian updating rule when more data become available. Before receiving additional data, however, the Bayesian approach remains a subjective representation of epistemic uncertainty.

2.1. Interval approach

This paper adopts the confidence interval approach to represent the unknown parameters θ. Let Θ denote the confidence intervals, and θ is a generic (arbitrary) element $\theta \in \Theta$. Under this assumption one needs to consider families of distributions whose parameters are in the intervals. Conceivably, the probability of failure p_f will not be unique and vary in an interval. We are interested in estimating a lower bound and an upper bound of p_f.

A visualization of all possible distributions with $\theta \in \Theta$ can be obtained by means of upper and lower distribution functions. Let $F(x)$ denote the cumulative distribution function (CDF) for the random variable X. For every x, an interval $[\underline{F}(x), \overline{F}(x)]$ generally can be readily found to bound the possible values of $F(x)$, i.e., $\underline{F}(x) \leqslant F(x) \leqslant \overline{F}(x)$, for $\theta \in \Theta$. Such a pair of two CDFs $\underline{F}(x)$ and $\overline{F}(x)$ construct a so-called *probability box* or *probability bounds* [6]. Fig. 1 shows the probability box for a normal distribution with an interval mean of [2.0, 3.0] and a standard deviation of 0.5. In this simple example, it is easy to verify that $\underline{F}(x)$ is the CDF of the normal variable with a mean of 3 and $\overline{F}(x)$ is the one with a mean of 2. Probability box represents a general framework to represent imprecisely specified distributions. It can represent not only distributions with unknown parameters, but also distributions with unknown type or even unknown dependencies [6].

Cartesian product method is routinely used for computing with probability boxes [6,7]. In this method, a probability box is discretized into a list of pairs $\{(A_1, m_1), \ldots, (A_i, m_i), \ldots\}$, in which A_i are intervals and m_i are their associated probability masses. A_i can be termed as focal elements and m_i can be viewed as the probability that A_i is the range of x [17]. Thus a probability box is analogous to a discrete probability distribution except that the probability mass is assigned to an interval rather than to a precise point. Different discretization methods have been proposed, such as the Outer Discretization Method and the Averaging Discretization Method [16]. The two discretization methods are graphically dem-

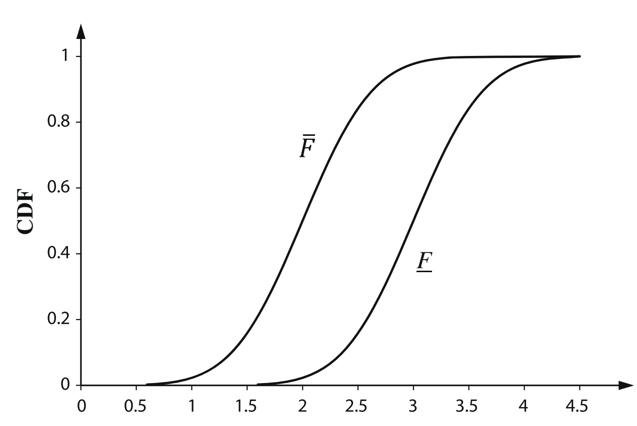

Fig. 1. A probability box defined by a normal distribution with a mean of [2.0, 3.0] and a standard deviation of 0.5.

onstrated in Fig. 2a and b. In practice, unbounded distributions are truncated to a finite range. Clearly, the accuracy of the approximation depends on how fine the discretization is [7].

Consider the limit state function $G(\mathbf{X})$ in Eq. (1). Suppose that the basic variables $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ are represented by n probability boxes. If the basic variables are statistically independent, the random relation pertaining to $(X_1, \ldots, X_n)$ can be defined by a probability box $\{(R_1, \rho_1), \ldots, (R_l, \rho_l)\}$ on the Cartesian product of the focal elements of $X_1 \times \cdots \times X_n$ [17,18]. The lower and upper bound on the probability of $G(\mathbf{X}) \leqslant 0$ can be evaluated as [4]

$$\bar{p}_f = \sum_{R_i : 0 \geqslant \inf(G(R_i))} \rho_i \tag{3}$$

$$\underline{p}_f = \sum_{R_i : 0 \geqslant \sup(G(R_i))} \rho_i \tag{4}$$

in which inf( ) and sup( ) denote infimum and supremum of the function, respectively. $\bar{p}_f$ and $\underline{p}_f$ represent an upper and lower bound for p_f respectively.

The above computation procedure requires that the image of every focal element R_i through the limit state function $G(\ )$ be calculated. If each probability box X_i has k focal elements, the total number of focal elements from the Cartesian products $X_1 \times \cdots \times X_n$ is k^n. For realistic engineering problems with large number of n and/or k, the computing effort of performing k^n structural analyses is prohibitive. Williamson [7] introduced a condensation strategy of constructing coarser discretization for probability boxes to reduce the calculation number to $(n-1)k^2$. However, there is a trade off between computational cost and accuracy of results. To overcome the difficulty associated with the Cartesian product method, an interval Monte Carlo simulation procedure has been developed.

3. Interval Monte Carlo simulation

3.1. Basic formula

In Monte Carlo simulation, the probability of failure is approximated as [1]

$$p_f \approx \frac{1}{N} \sum_{j=1}^{N} I[G(\hat{\mathbf{x}}_j) \leqslant 0] \tag{5}$$

where N is the total number of simulations conducted, $\hat{\mathbf{x}}_j$ represents the jth randomly simulated vector of basic variables, and $I[\ ]$ is the indicator function, having the value 1 if [ ] is 'true' and the value 0 if [ ] is 'false'.

The basic Monte Carlo simulation formula can be extended to the case when $f_{\mathbf{X}}(\mathbf{x})$ is a probability box with $\theta \in \Theta$. When θ varies in intervals, the randomly simulated basic variables $\hat{\mathbf{x}}_j$ vary in intervals accordingly. The limit state function $G(\hat{\mathbf{x}}_j)$ becomes a function of θ as well, i.e., $G(\hat{\mathbf{x}}_j, \theta)$. If the minimum and the maximum values of $G(\hat{\mathbf{x}}_j, \theta)$ can be determined

$$\mathrm{Min}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant G(\hat{\mathbf{x}}_j, \theta) \leqslant \mathrm{Max}(G(\hat{\mathbf{x}}_j, \theta)), \quad \text{for } \theta \in \Theta \tag{6}$$

then

$$I[\mathrm{Max}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0] \leqslant I[G(\hat{\mathbf{x}}_j, \theta) \leqslant 0] \leqslant I[\mathrm{Min}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0]. \tag{7}$$

Applying Eq. (7) in (5) gives

$$\frac{1}{N} \sum_{j=1}^{N} I[\mathrm{Max}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0] \leqslant \frac{1}{N} \sum_{j=1}^{N} I[G(\hat{\mathbf{x}}_j, \theta) \leqslant 0] \leqslant \frac{1}{N} \sum_{j=1}^{N} I[\mathrm{Min}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0]. \tag{8}$$

Thus, Eq. (8) provides an interval estimate for p_f

$$\underline{p}_f \approx \frac{1}{N} \sum_{j=1}^{N} I[\mathrm{Max}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0], \quad \bar{p}_f \approx \frac{1}{N} \sum_{j=1}^{N} I[\mathrm{Min}(G(\hat{\mathbf{x}}_j, \theta)) \leqslant 0], \quad \text{for } \theta \in \Theta. \tag{9}$$

3.2. Computational aspects

The first step in the implementation of interval Monte Carlo simulation is the generation of intervals in accordance with the prescribed probability boxes. The inverse transform method [3] is often used to generate random numbers. Consider a random variable X with CDF $F(x)$. If $(u_1, u_2, \ldots, u_m)$ is a set of values from the standard uniform variate, then the set of values

$$x_i = F_X^{-1}(u_i); \ i = 1, 2, \ldots, m \tag{10}$$

will have the desired CDF $F(x)$. The inverse transform method can be extended to perform random sampling from a probability box. Suppose that an imprecise CDF $F(x)$ is bounded by $\bar{F}(x)$ and $\underline{F}(x)$, as shown in Fig. 3. For each u_i in Eq. (10), two random numbers are generated

$$\underline{x}_i = \bar{F}^{-1}(u_i), \ \bar{x}_i = \underline{F}^{-1}(u_i). \tag{11}$$

Such a pair of $\underline{x}_i$ and $\bar{x}_i$ form an interval $[\underline{x}_i, \bar{x}_i]$ which contains all possible simulated numbers from the ensemble of distributions for a

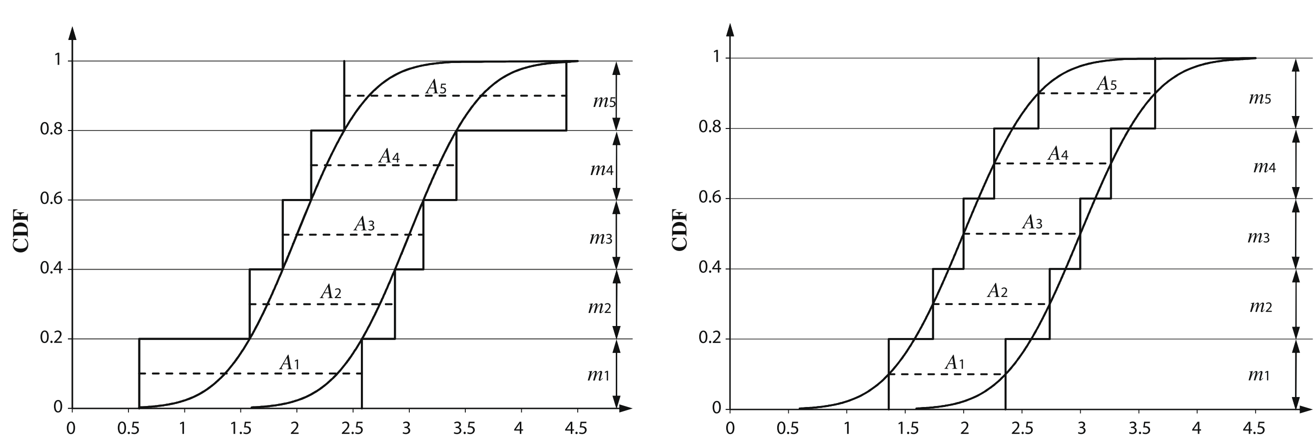

Fig. 2. Discretization of a probability box. (a) Outer Discretization Method; (b) Averaging Discretization Method [16].

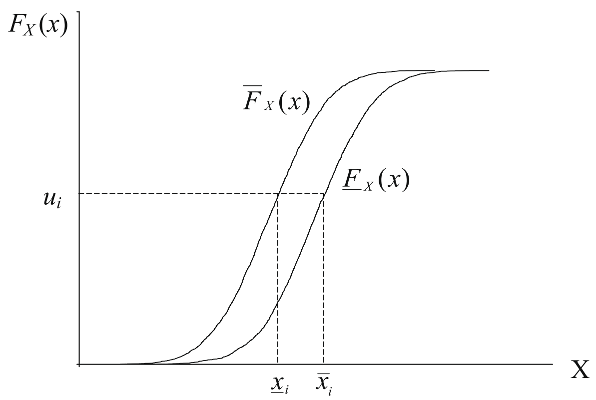

Fig. 3. Generation of random number from distribution with interval parameters.

particular u_i. The method is graphically demonstrated in Fig. 3 for one-dimensional case.

3.3. Computing the ranges of structure responses – interval FEM

The computational effort of the interval Monte Carlo simulation is contingent on the efficiency of computing the range (max. and min.) of structural responses through FE analyses when the simulated basic variables vary in intervals. This task can be performed by using the interval FEM. A variety of solution techniques have been proposed for the interval FEM, including the combinatorial method [19,20], perturbation method [21,22], sensitivity analysis method [23], optimization method [24,25], and interval analysis method [26–28]. In this paper, the interval FE analysis is formulated as an interval analysis problem. The interval analysis and interval FE formulation is briefly introduced below. Further details are provided in the authors' previous work [29–31].

Interval analysis concerns the numerical computations involving interval numbers. The four elementary operations of real arithmetic, namely addition (+), subtraction ($-$), multiplication ($\times$) and division ($\div$) can be extended to intervals. Operations $\circ \in \{+,-,\times,\div\}$ over interval numbers $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined by the general rule [32,33]

$$\boldsymbol{x} \circ \boldsymbol{y} = [\min(x \circ y), \max(x \circ y)] \quad \text{for } \circ \in \{+,-,\times,\div\} \tag{12}$$

in which x and y denote generic elements $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$. Software and hardware support for interval computation are available (e.g., [34,35]).

For a real-valued function $f(x_1,\ldots,x_n)$, the *interval extension* of $f(\ )$ is obtained by replacing each real variable x_i by an interval variable $\boldsymbol{x}_i$ and each real operation by its corresponding interval arithmetic operation. From the fundamental property of *inclusion isotonicity* [32], the range of the function $f(x_1,\ldots,x_n)$ can be rigorously bounded by its interval extension function

$$\{f(x_1,\ldots,x_n) \mid x_1 \in \boldsymbol{x}_1,\ldots,x_n \in \boldsymbol{x}_n\} \subseteq f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n). \tag{13}$$

Eq. (13) indicates that $f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$ contains the range of $f(x_1,\ldots,x_n)$ for all $x_i \in \boldsymbol{x}_i$.

A natural idea to implement interval FEM is to apply the interval extension to the deterministic FE formulation. Unfortunately, such a naïve use of interval analysis in FEM yields meaningless and overly wide results [27,28]. The reason is that in interval arithmetic each occurrence of an interval variable is treated as a *different, independent* variable. It is critical to the formulation of the interval FEM to identify the dependence between the interval variables and prevent the widening of results. In this paper, an element-by-element (EBE) technique is utilized for element assembly [27,30]. The elements are detached so that there are no connections between elements, avoiding element coupling. The penalty method is then employed to impose constraints to recover the connections between elements, and to ensure the compatibility of the displacements. The system equation in the interval FEM takes the following form

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p} \tag{14}$$

where $\boldsymbol{K}$ is the interval stiffness matrix, $\boldsymbol{u}$ is the interval displacement vector, $\boldsymbol{p}$ is the interval load vector, and Q is the deterministic penalty matrix. Eq. (14) can be transformed into a fixed point equation

$$R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0 + (I - R(\boldsymbol{K} + Q))\boldsymbol{u}^* = \boldsymbol{u}^* \tag{15}$$

in which R is a nonsingular preconditioning matrix, and u_0 is an approximate deterministic solution. It can be verified that $\boldsymbol{u} = \boldsymbol{u}^* + u_0$. Based on Eq. (15), interval fixed point iteration is constructed [33,36]

$$\boldsymbol{Z} + \boldsymbol{C}\boldsymbol{u}^{*(l)} = \boldsymbol{u}^{*(l+1)} \tag{16}$$

with $\boldsymbol{Z} = R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0$, and the iterative matrix $\boldsymbol{C} = I - R(\boldsymbol{K} + Q)$. The iteration converges when

$$\boldsymbol{u}^{*(l+1)} \subseteq \boldsymbol{u}^{*(l)}. \tag{17}$$

Then $\boldsymbol{u}^{*(l+1)} + u_0$ guarantees to contain the exact solution set of Eq. (14). The original interval fixed point iteration implicitly assumes that the coefficients of $\boldsymbol{K}$ vary independently between their bounds. This assumption is not valid for the system equations that arise in the interval FEM. Special formulation has to be developed to remove coefficient-dependence in the algorithm. By using the EBE technique, it is possible to decompose the interval stiffness matrix $\boldsymbol{K}$ into two parts

$$\boldsymbol{K} = S\boldsymbol{D} \tag{18}$$

in which S is a deterministic matrix and $\boldsymbol{D}$ is an interval diagonal matrix whose diagonal entries are the interval variables associated with each element (e.g., interval modulus of elasticity). The term $\boldsymbol{Z}$ in Eq. (16) can then be reintroduced as

$$\begin{aligned}\boldsymbol{Z} &= R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0 = R\boldsymbol{p} - RQu_0 - RS\boldsymbol{D}u_0 \\ &= R\boldsymbol{p} - RQu_0 - RSM\boldsymbol{\delta}.\end{aligned} \tag{19}$$

It must be noted that in Eq. (19) $\boldsymbol{D}u_0$ is introduced as $M\boldsymbol{\delta}$ in which M is a deterministic matrix and $\boldsymbol{\delta}$ is an interval vector [26]. The components of $\boldsymbol{\delta}$ are the diagonal entries of $\boldsymbol{D}$ with the difference that every interval variable occurs only once in $\boldsymbol{\delta}$. This treatment eliminates the coefficient-dependence in $\boldsymbol{Z}$, which is critical for obtaining a tight bound.

The interval fixed point iteration converges if and only if $\rho(|\boldsymbol{C}|) < 1$ [37], where $\rho(|\boldsymbol{C}|)$ is the spectral radius of the absolute value of the iterative matrix $\boldsymbol{C}$. To achieve a small $\rho(|\boldsymbol{C}|)$, the choice $R = (Q+S)^{-1}$ is made such that

$$\boldsymbol{C} = I - R(Q + S\boldsymbol{D}) = I - R(Q+S) - RS(\boldsymbol{D} - I) = -RS(\boldsymbol{D} - I). \tag{20}$$

Numerical tests have shown that fast convergence (within 10 iterations) generally can be achieved by using the above modified iterative algorithm. The developed linear elastic interval FEM has been successfully applied to plane frame structures, as well as continuum mechanics problems [29–31]. The structural responses can be accurately and efficiently computed.

4. Example 1: truss structure

Fig. 4 shows a planar truss. The serviceability limit state of deflection is considered. The deflection limit at the midspan is set to be 7.5 cm. Linear elastic analyses were performed.

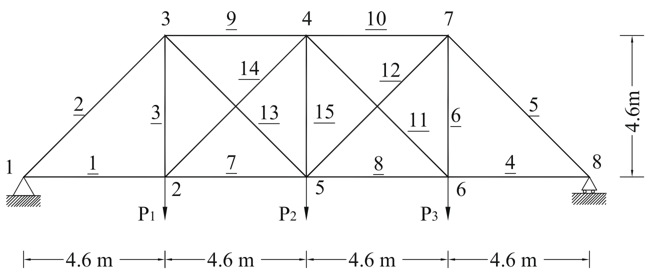

Fig. 4. Truss structure.

Table 1
Sample statistics for the basic random variables (truss in Fig. 4).

Variables	Sample mean	Sample standard deviation	No. samples
$A_1 - A_6$ (cm^2)	10.32	0.516	30
$A_7 - A_{15}$ (cm^2)	6.45	0.323	30
Ln P_1	4.483	0.09975	20
Ln P_2	5.582	0.09975	20
Ln P_3	4.483	0.09975	20

Unit of P: kN.

The cross-sectional areas for the 15 members and the three loads are identified as the basic random variables. All the 18 random variables are assumed to be mutually statistically independent. Assume that based on experience, the cross-sectional areas can be modeled by normals, and the loads modeled by lognormals. Suppose the statistics for the random variables were estimated from limited samples of data. Table 1 gives the available sample statistics for the cross-sectional areas and the logarithm of the loads (Ln P). The Young's modulus is assumed deterministic (200 GPa). The limit state probability is found to be 0.14% if the sample statistics are used. We now take into account the effects of parameter uncertainty on reliability assessment. Two cases are considered: (1) the uncertain means for the cross-sectional areas, and (2) the uncertain means for the (logarithm of) loads. The interval Monte Carlo method and the Bayesian approach are used to obtain interval estimates for the failure probability. The results from the two approaches are compared.

4.1. Case 1

4.1.1. Bayesian approach

Assume that the standard deviations of the cross-sectional areas are equal to those obtained from the samples. Let $N(\mu, \sigma)$ denote a normal distribution with a mean of μ and a standard deviation of σ. Assuming a uniform prior, for samples of size 30 the mean cross-sectional area of $A_1 - A_6$ and $A_7 - A_{15}$ are modeled by $N(10.32, 0.516/\sqrt{30})$ and $N(6.45, 0.323/\sqrt{30})$ respectively [3]. Sample statistics for the loads are used in this case.

Table 2
Bayesian confidence intervals for p_f, Case 1 (truss in Fig. 4).

Confidence	90%	95%	99%
p_f	[0.08%, 0.20%]	[0.063%, 0.22%]	[0.014%, 0.27%]

The distribution of p_f was numerically determined through a double-loop simulation procedure. In the outer loop, the means of cross-sectional areas were sampled. With the simulated mean values, the system failure probability was computed in the inner loop. 10,000 simulations were made for the mean cross-sectional areas. The relative frequency distribution of p_f is shown in Fig. 5 with a mean of 0.14% and a COV (coefficient of variation) of 0.3. It can be seen that the distribution of p_f is skewed. A lognormal is a good fit to the simulated data of p_f. Based on the simulated distribution of p_f, one can determine the Bayesian confidence interval for p_f corresponding to a specified probability. For example, one can state the probability that p_f will be in the interval from 0.08% to 0.2% is 0.9. Table 2 summarizes the intervals for p_f with different confidence. The Bayesian confidence intervals were determined such that the central point of the interval equals to the mean.

4.1.2. Interval approach

Table 3 summarizes the confidence intervals for the mean cross-sectional areas at different confidence levels. The random cross-sectional areas with interval means are bounded by probability boxes. For each confidence level, 500,000 interval Monte Carlo simulations were performed with randomly generated intervals for the cross-sectional areas according to their probability bounds. Interval FEM was utilized to find the minimum and the maximum of the deflection at the midspan. The last row of Table 3 gives the interval estimates of p_f, corresponding to the mean cross-sectional areas at a particular confidence level. For example, if the 95% confidence intervals are used to bound the mean cross-sectional areas, the failure probability is found to be between 0.048% and 0.30%. As expected, the width of the interval estimate for p_f becomes wider when a higher confidence level is used for the unknown parameters.

The computational efficiency of the proposed interval Monte Carlo simulation is evident in this example when compared with

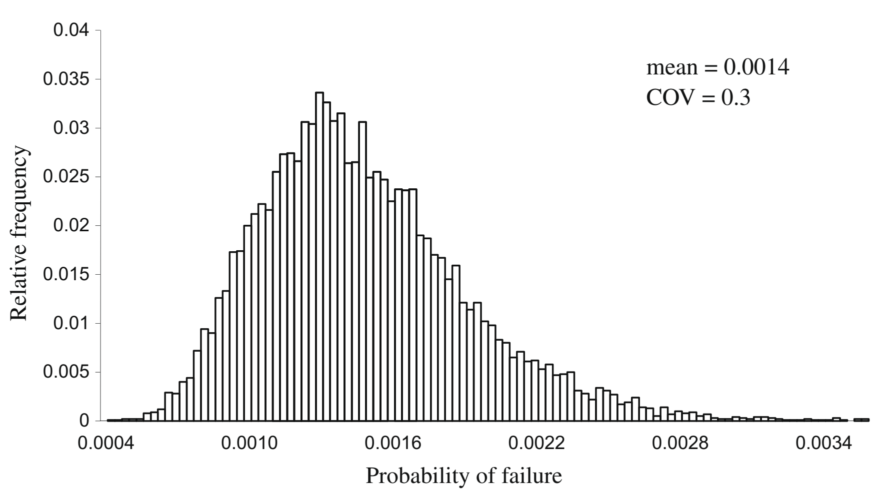

Fig. 5. Relative frequency distribution for p_f obtained from the Bayesian approach, Case 1 (truss in Fig. 4).

Table 3
Interval estimates for p_f obtained from interval Monte Carlo simulations, Case 1 (truss in Fig. 4).

Variables	Confidence level for the mean cross-sectional area		
	90%	95%	99%
$A_1 - A_6$ (cm^2)	[10.17, 10.48]	[10.14, 10.51]	[10.08, 10.56]
$A_7 - A_{15}$ (cm^2)	[6.35, 6.55]	[6.34, 6.57]	[6.30, 6.60]
p_f	[0.062%, 0.26%]	[0.048%, 0.30%]	[0.031%, 0.43%]

Table 4
Interval estimates for p_f obtained from the interval Monte Carlo simulations, Case 2 (truss in Fig. 4).

Variables	Confidence level for λ_i		
	90%	95%	99%
λ_1, λ_3	[4.4465, 4.5199]	[4.4395, 4.5269]	[4.4258, 4.5407]
λ_2	[5.5452, 5.6186]	[5.5381, 5.6256]	[5.5244, 5.6393]
p_f	[0.025%, 0.67%]	[0.019%, 0.72%]	[0.01%, 1.18%]

the Cartesian product method. The Cartesian product method would require 10^{15} analyses if each of the 15 probability boxes is discretized into 10 focal elements. Such a computing effort makes the Cartesian product method impractical in this example.

4.2. Case 2

4.2.1. Bayesian approach

The natural logarithm of loads ($\mathrm{Ln}\,P_i, i = 1, 2, 3$) are normal distributions. Assume that the standard deviations for Ln P_i are equal to their sample standard deviations. Let λ_i denote the mean of Ln P_i. According to the Bayesian approach, with uniform priors the posterior distributions of λ_1 and λ_3 are $N(4.483, 0.09975/\sqrt{20})$, and λ_2 is $N(5.582, 0.09975/\sqrt{20})$. 10,000 samples were drawn for each λ_i. The system failure probability associated with the sampled λ_i was computed and the frequency distribution for p_f is shown in Fig. 6. It is evident that the frequency distribution is broad and skewed to the right with a long upper tail. The mean value of p_f is 0.17% with a very high COV of 0.76. The long upper tail and high COV of p_f suggests that the parameters λ_i have a significant effect on the failure reliability. The results shown in Fig. 6 also suggest that the system failure probability is between 0.017% and 0.34% with 90% confidence.

4.2.2. Interval approach

Table 4 presents the confidence intervals on λ_i and the corresponding interval estimates on the failure probability. 500,000 interval Monte Carlo simulations were performed. Comparing Tables 3 and 4 shows that the widths of the interval failure probabilities are noticeably wider in Case 2 than in Case 1. This suggests that the failure probability is more sensitive to the errors in λ_i than the mean cross-sectional areas. As evidenced in Table 4, the interval failure probability only becomes slightly wider when the confidence level for λ_i increases from 90% to 95%. On the other hand, the interval p_f changes from [0.019%, 0.72%] to [0.01%, 1.18%] when the confidence level for λ_i increases from 95% to 99%. The high upper bound of the interval p_f corresponds to the long upper tail of the distribution of p_f obtained in the Bayesian approach.

4.3. Comparison of Bayesian approach and interval approach

In this example the Bayesian approach and the interval approach both imply that the mean values of (logarithm of) loads and member cross-sectional areas have significant effects on the structural reliability. The system reliability is particularly sensitive to the means of (logarithm of) loads. Both approaches suggest that additional data, particularly for loads, should be collected if more confidence in the reliability estimate is needed.

Although the Bayesian approach and the interval approach provide comparable interval estimates for the failure probability, these two approaches are conceptually different. The results have different meanings. The interval estimate of p_f obtained from the Bayesian approach has an explicit statistical implication. Comparing Tables 2 and 3 indicates that the interval failure probabilities obtained from the interval approach tend to be wider than those of the Bayesian approach. This is mainly due to the fact that prior information on the unknown parameters was assumed and incorporated in the Bayesian approach. On the other hand, the interval approach only uses the information of the bounds on unknown parameters, thus does not provide a statistical statement on the resulting interval failure probability. Although some subjective judgment is still needed in the interval approach, such as selecting appropriate confidence levels for the unknown parameters, it generally requires less subjective information than the Bayesian approach. The authors make no judgment as to the relative validity of the Bayesian approach and the interval approach. The purpose here is to demonstrate the feasibility of propagating interval information on unknown parameters through reliability assessment by using the proposed interval Monte Carlo methods.

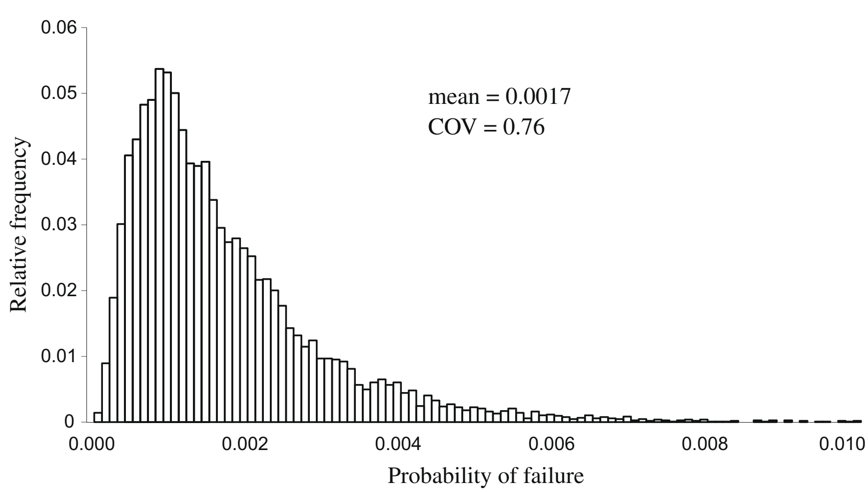

Fig. 6. Relative frequency histogram for p_f obtained from the Bayesian approach, Case 2 (truss in Fig. 4).

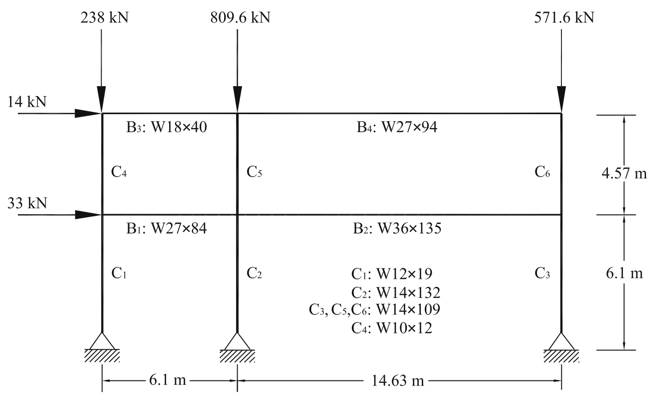

Fig. 7. Two-bay two-story frame.

5. Example 2: frame structure

The second example demonstrates the application of the developed method on frame structures. A two-bay two-story steel frame shown in Fig. 7 is considered. The frame geometry and member sizes are based on those of [38]. In the figure the column is denoted as 'C' and the beam as 'B'. Subscripts indicate member number. The frame is subjected to lateral loads at each floor and vertical loads at the top. The loads are assumed deterministic. The Young's modulus is 200 GPa. The sectional properties (moment of inertia I and cross-sectional area A) of each member are considered as basic random variables. Suppose only interval bound information is available for the means of the random variables. Table 5 summarizes the interval means and the standard deviations for the basic random variables. All random variables are assumed normals. For simplicity, it is assumed that the moment of inertia and cross-sectional area are statistically independent. Perfectly correlation is assumed between column-to-column, and between beam-to-beam. No correlation exists between column-to-beam.

The serviceability limit state considered is $H/500$ (21 mm) for roof drift, where H is the total height of the frame. Linear elastic analyses were performed. The bounds of the cumulative frequency distribution of the roof drift were obtained after 10,000 interval Monte Carlo simulations, and is as shown in Fig. 8. The probability of failure is found to be between 0.11% and 1.29%.

Table 5
Random sectional properties for the frame of Fig. 7.

Member	μ_I (cm^4)	σ_I (cm^4)	μ_A (cm^2)	σ_A (cm^2)
C_1	[5327.2, 5494.9]	270.55	[35.4,36.5]	1.80
C_2	[62696.6, 64670.2]	3184.2	[246.4,254.2]	12.5
C_4	[2204.6, 2274.0]	111.97	[22.5,23.2]	1.14
C_3, C_5, C_6	[50813.0,52412.4]	2580.6	[203.3, 209.7]	10.32
B_1	[116787.9, 120464.1]	5931.3	[157.5,162.5]	8.0
B_2	[319629.9, 329691.1]	16233.0	[252.2,260.1]	12.81
B_3	[25078.7, 25868.1]	1273.7	[75.0,77.3]	3.81
B_4	[133998.7, 138216.6]	6805.4	[176.0,181.5]	8.94

μ: mean; σ: standard deviation.

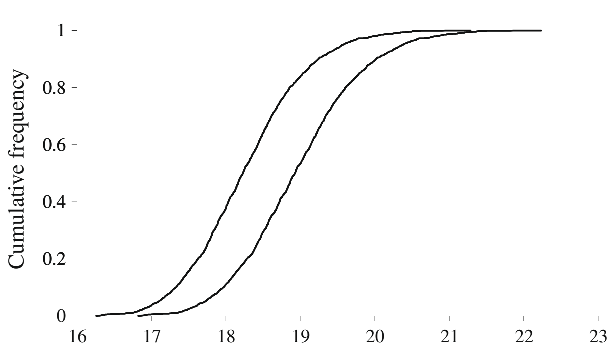

Fig. 8. Bounds of the cumulative frequency distribution of the roof drift (frame in Fig. 7).

6. Conclusion

An interval Monte Carlo method has been developed for reliability assessment under parameter uncertainties represented by confidence intervals. The interval information of unknown parameters and the inherent uncertainties are propagated separately through reliability analysis. Interval FEM is utilized to model the ranges of structural responses accurately and efficiently. Interval estimates of failure probability are computed which can provide a statement of confidence in the results of the reliability estimate. A wide interval p_f implies that epistemic uncertainties are large, thus additional data should be collected. The developed method can also be used to study the sensitivities of failure probability with respect to the changes in distribution parameters.

A truss structure and a steel frame have been analyzed to illustrate the proposed method. In the truss example, the results produced by using the proposed method and the Bayesian approach were compared. Both approaches can obtain interval estimates on the limit state probability and provide decision-makers and designers useful information about the variabilities in reliability estimates. The interval p_f obtained from the proposed method tends to be wider than that from the Bayesian approach. This is to be expected since the interval approach only uses the information of the confidence bounds on the unknown parameters, while in the Bayesian approach prior information is assumed and incorporated in analysis.

The introduced interval Monte Carlo method is based on direct sampling. Future work can explore the possibility of extending the interval analysis with more advanced sampling methods such as the importance sampling to improve the computational efficiency.

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